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The Conservation Equations of the Charge and Current Densities of a Free Relativistic Electron

T. L. CHOW

California State College, Stanislaus, Turlock, California 95380

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Abstract

The four-current density j_μ was first decomposed into convection current part and spin current part, respectively. We then demonstrate that the convection current density and spin current density are independently conserved.

It is well known that the four-current density (charge-current density) j_μ of a free relativistic electron can be decomposed into two parts by Gordon decomposition: the convection part and the spin part. It is the purpose of this note to show that the convection current density and spin current density are independently conserved.

The four-current density j_μ is given by $iec\bar{\psi}\gamma_\mu\psi$, and it can be split into two parts (Sakurai, 1967):

$$j_\mu = j_\mu^{(1)} + j_\mu^{(2)}$$

where

$$j_\mu^{(1)} = \frac{ie\hbar}{2m_0}\left(\frac{\partial\bar{\psi}}{\partial x_\mu}\psi - \bar{\psi}\frac{\partial\psi}{\partial x_\mu}\right),$$

$$j_\mu^{(2)} = \frac{ie\hbar}{2m_0}\left(\frac{\partial\bar{\psi}}{\partial x_\nu}\gamma_\nu\gamma_\mu\psi - \bar{\psi}\gamma_\mu\gamma_\nu\frac{\partial\psi}{\partial x_\mu}\right)_{\mu\neq\nu}$$

here and hereafter we follow Sakurai's notation.

In terms of matrices $\boldsymbol{\alpha}$ and β, which relate to the gamma matrices by

$$\gamma_k = -i\beta\alpha_k, \qquad \gamma_4 = \beta, \qquad k = 1, 2, 3$$

$j_\mu^{(1)}$ and $j_\mu^{(2)}$ becomes

$$\rho = \frac{j_4^{(1)} + j_4^{(2)}}{ic} = \frac{ie\hbar}{2m_0c^2}\left(\psi^+\beta\frac{\partial\psi}{\partial t} - \frac{\partial\psi^+}{\partial t}\beta\psi\right) + \frac{ie\hbar}{2m_0c}(\psi^+\beta\boldsymbol{\alpha}.\nabla\psi - \nabla\psi^+)$$

and

$$\mathbf{j} = \frac{e\hbar}{2m_0}\left[\frac{1}{i}(\psi^+\beta\nabla\psi - \nabla\psi^+\beta\psi) + \text{curl}\,(\psi^+\sigma\beta\psi) + \frac{1}{c}\frac{\partial}{\partial t}\left(\psi^+\frac{\boldsymbol{\alpha}\beta}{i}\psi\right)\right]$$

where ρ and $\mathbf{j}$ are the charge density and current density, respectively. The first term on the right-hand side of either ρ or $\mathbf{j}$ is due to the convection current, and it differs from the corresponding Klein–Gordon expression by the presence of the factor β:

$$\rho^{\text{con}} = \frac{ie\hbar}{2m_0c^2}\left(\psi^+\beta\frac{\partial\psi}{\partial t} - \frac{\partial\psi^+}{\partial t}\beta\psi\right)$$

$$\mathbf{j}^{\text{con}} = -\frac{ie\hbar}{2m_0}(\psi^+\beta\nabla\psi - \nabla\psi^+\beta\psi)$$

The other two terms on the right-hand side of either ρ or $\mathbf{j}$ are due to spin contribution:

$$\rho^{\text{spin}} = \frac{ie\hbar}{2m_0c}(\psi^+\beta\boldsymbol{\alpha}.\nabla\psi - \nabla\psi^+.\boldsymbol{\alpha}\beta\psi)$$

$$\mathbf{j}^{\text{spin}} = c\,\text{curl}\left(\frac{e\hbar}{2m_0c}\psi^+\sigma\beta\psi\right) + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{e\hbar}{2m_0i}\psi^+\boldsymbol{\alpha}\beta\psi\right)$$

According to classical electrodynamics, the magnetization current density of a magnetization vector (magnetic dipole moment density) $\mathbf{M}$ is $c\,(\text{curl}\,M)$ (Panofsky *et al.*, 1962), hence the quantity $(e\hbar/2m_0c)\psi^+\sigma\beta\psi$ in the first term of $\mathbf{j}^{\text{spin}}$ is the magnetic dipole moment density $\mathbf{M}$ of the electron. Again, according to classical electrodynamics, the polarization current of a polarization vector (electric dipole moment density) $\mathbf{P}$ is $(1/c)(\partial\mathbf{P}/\partial t)$. Hence, the second term of $\mathbf{j}^{\text{spin}}$ is the polarization current associated with the electric dipole moment density $\mathbf{P}$ $[= (e\hbar/2m_0i)\psi^+\beta\boldsymbol{\alpha}\psi]$ of the electron. In terms of $\mathbf{M}$ and $\mathbf{P}$, $\mathbf{j}^{\text{spin}}$ can be written

$$\mathbf{j}^{\text{spin}} = \mathbf{j}^{\text{con}} + c(\text{curl}\,\mathbf{M}) + \frac{1}{c}\frac{\partial\mathbf{P}}{\partial t}$$

where

$$\mathbf{M} = \frac{e\hbar}{2m_0c}\psi^+\sigma\beta\psi$$

$$\mathbf{P} = \frac{e\hbar}{2m_0i}\psi^+\beta\boldsymbol{\alpha}\psi$$

It is of interest to note that one can demonstrate directly the spin magnetic moment of the electron. As **M** is the magnetic dipole moment density, the total magnetic dipole moment of the electron is therefore given by

$$\int \mathbf{M}\, d\tau = \frac{e\hbar}{2m_0 c} \int \psi^+ \sigma \beta \psi \, d\tau$$

and the average value of β is $\sqrt{[1-(v^2/c^2)]}$ (Chow, in preparation). Thus, the spin magnetic moment of a relativistic free electron is $e\hbar/2mc$, where m is the Lorentz mass.

Finally, it is easy to show that ρ^{con} and $\mathbf{j}^{\text{con}}$, or ρ^{spin} and $\mathbf{j}^{\text{spin}}$ are independently conserved. Taking the time derivative of ρ^{con}, we obtain

$$\frac{\partial \rho^{\text{con}}}{\partial t} = \frac{ie\hbar}{2m_0 c^2}\left(\psi^+ \beta \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi^+}{\partial t^2} \beta \psi\right)$$

With the help of the Klein-Gordon equation, the above equation reduces to

$$\frac{\partial \rho^{\text{con}}}{\partial t} = \frac{ie\hbar}{2m_0} (\psi^+ \beta \nabla^2 \psi - \nabla \psi^+ \beta \psi)$$

It is not difficult to recognize that the right-hand side of the above equation is just equal to $-\text{div}(\mathbf{j}^{\text{con}})$. Thus,

$$\text{div}(\mathbf{j}^{\text{con}}) + \frac{\partial \rho^{\text{con}}}{\partial t} = 0$$

Then as the total current density and charge density satisfy the conservation equation $\text{div}(\mathbf{j}) + (\partial \rho/\partial t) = 0$, $\mathbf{j}^{\text{spin}}$ and ρ^{spin} are therefore independently conserved:

$$\text{div}(\mathbf{j}^{\text{spin}}) + \frac{\partial \rho^{\text{spin}}}{\partial t} = 0$$

References

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